ROLLING OF A SPHERE ON AN INCLINED PLANE

(KACHENIE SHARA PO NAKLONNOI PLOSKOSTI)

PMM Vol.22, No.4, 1958, pp.504-509

E. I. KHARLAMOVA
(Moscow)

(Received 20 March 1958)

The problem of a sphere rolling on a horizontal plane was completely solved by Chaplygin on the assumption that the center of gravity of the sphere coincides with its geometric center, [centroid]. As far as we know, the problem of a nonhomogeneous sphere rolling on an inclined plane, having a triaxial ellipsoid of inertia about the center of gravity which coincides with the geometric center of the sphere, has not yet been solved. We show in this paper that if certain restrictions are imposed on the initial conditions, the equations of motion of a sphere on an inclined plane can be reduced to equations whose form is identical with those investigated by Chaplygin.

With the inclined plane we associate a coordinate system $O_1\xi_1\eta_1\zeta_1$, where the axis $O_1\zeta_1$ is directed normal to the plane and $O_1\xi_1$ is directed along the line of steepest descent. We take the origin of the coordinate system $O\xi\eta\zeta$, with axes parallel to $O_1\xi_1\eta_1\zeta_1$, at the geometric center of the sphere.

Let R_ξ, R_η, R_ζ be the components, along the axes $O\xi\eta\zeta$, of the reaction of the plane at the point of contact of the sphere, and let K_ξ, K_η, K_ζ be the projections of the angular momentum of the sphere relative to its center onto the same axes. We denote by $v(v_\xi, v_\eta, 0)$ the velocity of the center of the sphere and by $\omega(\omega_\xi, \omega_\eta, \omega_\zeta)$ its angular velocity.

Assuming that the center of gravity of the sphere coincides with its geometric center, the general equations of dynamics are

$$m\frac{dv_\xi}{dt} = F + R_\xi, \qquad m\frac{dv_\eta}{dt} = R_\eta$$

$$\frac{dK_\xi}{dt} = \rho R_\eta, \qquad \frac{dK_\eta}{dt} = -\rho R_\xi, \qquad \frac{dK_\zeta}{dt} = 0 \tag{1}$$

where m is the mass of the sphere, ρ is its radius, and F is the component of gravity parallel to the plane.

From (1) we obtain

$$\frac{dK_\xi}{dt} = m\rho\frac{dv_\eta}{dt}, \quad \frac{dK_\eta}{dt} = -\rho\left(m\frac{dv_\xi}{dt} - F\right), \quad \frac{dK_\zeta}{dt} = 0$$

or

$$K_\xi - m\rho v_\eta = n, \quad K_\eta + m\rho v_\xi = \rho F t, \quad K_\zeta = h$$

We have set the constant of the second integral equal to zero. There is no loss in generality since the time does not occur explicitly in the corresponding differential equation.

We suppose that the sphere rolls without sliding and that therefore the points of contact of the sphere with the plane have zero velocity. Then

$$\mathbf{v} - \boldsymbol{\omega}\times\rho\mathbf{k} = 0 \tag{2}$$

where $\mathbf{k}$ is the unit vector along $O\zeta$. Projecting on the axes $O\xi\eta\zeta$, we obtain

$$v_\xi = \rho\omega_\eta, \qquad v_\eta = -\rho\omega_\xi \tag{3}$$

Hence

$$K_\xi + m\rho^2\omega_\xi = n, \quad K_\eta + m\rho^2\omega_\eta = \rho F t, \quad K_\zeta = h \tag{4}$$

With the sphere we associate a fixed coordinate system $Oxyz$, with axes having the same directions as the principal axes of the ellipsoid of inertia at its center O. We denote by L, M, N the principal moments of inertia about the center, and the components of the angular velocity along these axes by p, q, r. If the projections of the unit vectors along the axes $O\xi$, $O\eta$, $O\zeta$ on the axes $Oxyz$ are α, α', α'', β, β', β'', γ, γ', γ'', respectively, then

$$\begin{aligned} \omega_\xi &= p\alpha + q\alpha' + r\alpha'', \qquad & K_\xi &= Lp\alpha + Mq\alpha' + Nr\alpha'', \\ \omega_\eta &= p\beta + q\beta' + r\beta'', & K_\eta &= Lp\beta + Mq\beta' + Nr\beta'' \\ \omega_\zeta &= p\gamma + q\gamma' + r\gamma'', & K_\zeta &= Lp\gamma + Mq\gamma' + Nr\gamma'' \end{aligned} \tag{5}$$

From (5) we obtain

$$(K_\xi + m\rho^2\omega_\xi)^2 + (K_\eta + m\rho^2\omega_\eta)^2 + (K_\zeta + m\rho^2\omega_\zeta)^2 = n^2 + \rho^2F^2t^2 + (h + m\rho^2\omega_\zeta)^2 \tag{6}$$

or, in terms of the axes $Oxyz$,

$$\begin{aligned} &(L + m\rho^2)^2p^2 + (M + m\rho^2)^2q^2 + (N + m\rho^2)^2r^2 = \\ &= n^2 + \rho^2F^2t^2 + [h + m\rho^2(p\gamma + q\gamma' + r\gamma'')]^2 \end{aligned} \tag{7}$$

Since the unit vector $\mathbf{k}(\gamma, \gamma', \gamma'')$ is invariant, taking (2) into account, we have

$$\frac{d\mathbf{k}}{dt} + \boldsymbol{\omega}\times\mathbf{k} = 0, \qquad \mathbf{v} = -\rho\frac{d\mathbf{k}}{dt} \tag{8}$$

and consequently the square of the velocity may be expressed as

$$v^2 = \rho^2 (\dot{\gamma}^2 + \dot{\gamma}'^2 + \dot{\gamma}''^2) \tag{9}$$

Hence the energy integral can be written as

$$Lp^2 + Mq^2 + Nr^2 + m\rho^2 (\dot{\gamma}^2 + \dot{\gamma}'^2 + \dot{\gamma}''^2) = 2F\xi_1 + l \tag{10}$$

where l is a constant of the *vis viva* which, without loss of generality, can be made zero by the choice of the origin of the fixed coordinate system on the inclined plane.

In addition, $v_\xi = d\xi_1/dt$ and $v_\eta = d\eta_1/dt$. Hence, in view of (3) and (5),

$$\frac{d\xi_1}{dt} = \rho (p\beta + q\beta' + r\beta''), \qquad \frac{d\eta_1}{dt} = -\rho (p\alpha + q\alpha' + r\alpha'') \tag{11}$$

Equation (8) gives

$$\frac{d\gamma}{dt} = r\gamma' - q\gamma'', \qquad \frac{d\gamma'}{dt} = p\gamma'' - r\gamma, \qquad \frac{d\gamma''}{dt} = q\gamma - p\gamma' \tag{12}$$

From these equations and the third equation of (4), which by (5) can be written as

$$Lp\gamma + Mg\gamma' + Nr\gamma'' = h \tag{13}$$

we find

$$\begin{aligned} p (L\gamma^2 + M\gamma'^2 + N\gamma''^2) &= h\gamma + N\gamma''\dot{\gamma}' - M\gamma'\dot{\gamma}'' \\ q (L\gamma^2 + M\gamma'^2 + N\gamma''^2) &= h\gamma' + L\gamma\dot{\gamma}'' - N\gamma''\dot{\gamma} \\ r (L\gamma^2 + M\gamma'^2 + M\gamma''^2) &= h\gamma'' + M\gamma'\dot{\gamma} - L\gamma\dot{\gamma}' \end{aligned} \tag{14}$$

The second equation of (4) can be written as

$$(L + m\rho^2)\, p\beta + (M + m\rho^2)\, q\beta' + (N + m\rho^2)\, r\beta'' = \rho F t$$

If the formulas $\beta\gamma + \beta'\gamma' + \beta''\gamma'' = 0$, $\beta^2 + \beta'^2 + \beta''^2 = 1$ (the relations between the direction cosines relative to rectangular axes) are adjoined to (c) and if these three expressions (2) are resolved relative β, β', β'', we obtain, in view of (7) and (13),

$$\begin{aligned} (n^2 + \rho^2F^2t^2)\,\beta &= n\ [(M + m\rho^2)\, q\gamma'' - (N + m\rho^2)\, r\gamma'] + (L + m\rho^2)\, p\rho F t - \\ &\quad - \rho F t\, [h + m\rho^2 (p\gamma + q\gamma' + r\gamma'')]\,\gamma \\ (n^2 + \rho^2F^2t^2)\,\beta' &= n\ [(N + m\rho^2)\, r\gamma - (L + m\rho^2)\, p\gamma''] + (M + m\rho^2)\, q\rho F t - \\ &\quad - \rho F t\, [h + m\rho^2 (p\gamma + q\gamma' + r\gamma'')]\,\gamma' \\ (n^2 + \rho^2F^2t^2)\,\beta'' &= n\, [(L + m\rho^2)\, p\gamma' - (M + m\rho^2)\, q\gamma] + (N + m\rho^2)\, r\rho F t - \\ &\quad - \rho F t\, [h + m\rho^2 (\rho\gamma + q\gamma' + r\gamma'')]\,\gamma'' \end{aligned} \tag{15}$$

Substituting (15) into the first equation of (11), we get

$$\frac{n^2+\rho^2F^2t^2}{\rho}\frac{d\xi_1}{dt}=n\{[(M+m\rho^2)\,q\gamma''-(N+m\rho^2)\,r\gamma']\,p+$$
$$+[(N+m\rho^2)\,r\gamma-(L+m\rho^2)\,p\gamma'']\,q+[(L+m\rho^2)\,p\gamma'-(M+m\rho^2)\,q\gamma]\,r\}+$$
$$+\rho Ft\,[(L+m\rho^2)\,p^2+(M+m\rho^2)\,q^2+(N+m\rho^2)\,r^2]-$$
$$-\rho^2Ftm\,(p\gamma+q\gamma'+r\gamma'')^2-h\rho Ft\,(p\gamma+q\gamma'+r\gamma'') \qquad (16)$$

The expression included in the braces can be written as

$$Lp\,(r\gamma'-q\gamma'')+Mq\,(p\gamma''-r\gamma)+Nr\,(q\gamma-p\gamma')=Lp\dot\gamma+Mq\dot\gamma'+Nr\dot\gamma'' \quad (17)$$

The second and third terms on the right side of (16), in view of (10) and the obvious relation

$$\gamma^2+\gamma'^2+\gamma''^2=1,$$

are transformed into

$$\rho Ft\,\{Lp^2+Mq^2+Nr^2+m\rho^2\,[(p^2+q^2+r^2)\,(\gamma^2+\gamma'^2+\gamma''^2)-$$
$$-(p\gamma+q\gamma'+r\gamma'')^2]\}=\rho Ft\,\{Lp^2+Mq^2+Nr^2+$$
$$+m\rho^2\,[(r\gamma'-q\gamma'')^2+(p\gamma''-r\gamma)^2+(q\gamma-p\gamma')^2]\}=$$
$$=\rho Ft\,[Lp^2+Mq^2+Nr^2+m\rho^2\,(\dot\gamma^2+\dot\gamma'^2+\dot\gamma''^2)]=2\rho F^2\xi_1 t$$

Using (14), we find

$$p\gamma+q\gamma'+r\gamma''=\frac{1}{H}\,[(M-N)\,\gamma'\gamma''\dot\gamma+(N-L)\,\gamma''\gamma\dot\gamma'+(L-M)\,\gamma\gamma'\dot\gamma''+h] \quad (18)$$

where

$$H=L\gamma^2+M\gamma'^2+N\gamma''^2$$

Equation (16) can be written in the form

$$\frac{n^2+\rho^2F^2t^2}{\rho}\frac{d\xi_1}{dt}=n\,(Lp\dot\gamma+Mq\dot\gamma'+Nr\dot\gamma'')+2\rho F^2\xi_1 t-$$
$$-\frac{1}{H}\,\rho Fht\,[(M-N)\,\gamma'\gamma''\dot\gamma+(N-L)\,\gamma''\gamma\dot\gamma'+(L-M)\,\gamma\gamma'\dot\gamma''+h] \qquad (19)$$

Hence we have a closed system of equations (7), (10), (14), (17), (19) which determine p, q, r, γ, γ', γ'', ξ_1.

Eliminating p, q, r from (7) by means of (14), we get

$$MN\dot\gamma^2+NL\dot\gamma'^2+LM\dot\gamma''^2+h^2+$$
$$+(L\gamma^2+M\gamma'^2+N\gamma''^2)\,[m\rho^2\,(\dot\gamma^2+\dot\gamma'^2+\dot\gamma''^2)-2F\xi_1]=0 \qquad (20)$$

For convenience in the further calculations we write the integral (10) in the form

$$(L+m\rho^2)\,p^2+(M+m\rho^2)\,q^2+(N+m\rho^2)\,r^2=2F\xi_1+$$
$$+m\rho^2\,[(p^2+q^2+r^2)\,(\gamma^2+\gamma'^2+\gamma''^2)-(r\gamma'-q\gamma'')^2-(p\gamma''-r\gamma)^2-(q\gamma-p\gamma')^2]$$

or

$$(L+m\rho^2)p^2+(M+m\rho^2)q^2+(N+m\rho^2)r^2=2F\xi_1+m\rho^2(p\gamma+q\gamma'+r\gamma'')^2$$

Multiplying this expression by $m\rho^2$ and subtracting from the integral (7), we obtain

$$\begin{aligned}L(L+m\rho^2)p^2+M(M+m\rho^2)q^2+N(N+m\rho^2)r^2=\\=n^2+\rho^2F^2t^2+h^2+2hm\rho^2(p\gamma+q\gamma'+r\gamma'')-2m\rho^2F\xi_1\end{aligned}$$

In view of (14) and (18) this relation may be rewritten as

$$\begin{aligned}&\frac{1}{H^2}[(LN\gamma''\dot{\gamma}'-LM\gamma'\dot{\gamma}'')^2+(ML\gamma\dot{\gamma}''-MN\dot{\gamma}\gamma'')^2+(NM\gamma'\dot{\gamma}-NL\gamma\dot{\gamma}')^2]+\\&+\frac{m\rho^2}{H}(MN\dot{\gamma}^2+NL\dot{\gamma}'^2+LM\dot{\gamma}''^2)=-\frac{h^2}{H^2}(L^2\gamma^2+M^2\gamma'^2+N^2\gamma''^2)-\\&-\frac{2h}{H^2}[MN(N-M)\gamma'\gamma''\dot{\gamma}+NL(L-N)\gamma''\gamma\dot{\gamma}'+LM(M-L)\gamma\gamma'\dot{\gamma}'']-\\&+\frac{2h}{H}m\rho^2[(M-N)\gamma'\gamma''\dot{\gamma}+(N-L)\gamma''\gamma\dot{\gamma}'+(L-M)\gamma\gamma'\dot{\gamma}''+h]-\\&-\frac{m\rho^2h^2}{H}+h^2+n^2+\rho^2F^2t^2-2m\rho^2F\xi_1\end{aligned}\tag{21}$$

Eliminating p, q, r from (19) yields

$$\begin{aligned}&\frac{n^2+\rho^2F^2t^2}{\rho}\frac{d\xi_1}{dt}=2\rho F^2\xi_1t-\\&-\frac{\rho Fht}{H}[h+(M-N)\gamma'\gamma''\dot{\gamma}+(N-L)\gamma''\gamma\dot{\gamma}'+(L-M)\gamma\gamma'\dot{\gamma}'']+\\&+\frac{n}{H}[L(M-N)\gamma\dot{\gamma}'\dot{\gamma}''+M(N-L)\gamma'\dot{\gamma}''\dot{\gamma}+N(L-M)\gamma''\dot{\gamma}\dot{\gamma}']\end{aligned}\tag{22}$$

If the sphere is placed on the plane without applying an initial velocity, the constants of integration h and n in (4) are zero. In that case, (22) reduces to

$$\frac{d\xi_1}{dt}=2\frac{\xi_1}{t},\qquad\text{or}\qquad \xi_1=\frac{1}{2}\nu t^2\tag{23}$$

We shall determine the constant ν. The position of the sphere on the plane is determined by the coordinates ξ_1, η_1 of the center and the orientations of the axes $Oxyz$ relative to the plane, i.e. by the Euler angles. The initial position of the sphere on the plane determines the initial values of the Euler angles, and consequently the initial values α_0, α_0', α_0'', β_0, β_0', β_0'', γ_0 γ_0' γ_0'' of all nine direction cosines.

Since the sphere is initially at rest,

$$p_0 = q_0 = r_0 = 0 \tag{24}$$

Substituting (5) and (4), we get

$$\begin{aligned} &Ap\alpha + Bq\alpha' + Cr\alpha'' = 0 \\ &Ap\beta + Bq\beta' + Cr\beta'' = \rho F t \\ &Lp\gamma + Mq\gamma' + Nr\gamma'' = 0 \end{aligned} \tag{25}$$

where for brevity we have introduced the notation

$$A = L + m\rho^2 \qquad B = M + m\rho^2, \qquad C = N + m\rho^2 \tag{26}$$

Since $d\xi_1 / dt = \nu t$, the first expression of (11) gives

$$p\beta + q\beta' + r\beta'' = \frac{\nu}{\rho} t \tag{27}$$

Differentiating (25) and (27) with respect to t and setting $t = 0$, we get, by (24),

$$A\alpha_0 \dot{p}_0 + B\alpha_0' \dot{q}_0 + C\alpha_0'' \dot{r}_0 = 0$$

$$A\beta_0 \dot{p}_0 + B\beta_0' \dot{q}_0 + C\beta_0'' \dot{r}_0 = \rho F, \qquad L\gamma_0 \dot{p}_0 + M\gamma_0' \dot{q}_0 + N\gamma_0'' \dot{r}_0 = 0$$

$$\beta_0 \dot{p}_0 + \beta_0' \dot{q}_0 + \beta_0'' \dot{r}_0 = \frac{\alpha}{\rho} \tag{28}$$

The determinant of the first three equations relative to the required initial values $\dot{p}_0$, $\dot{q}_0$, $\dot{r}_0$ is different from zero. Indeed,

$$\Delta = \begin{vmatrix} A\alpha_0 & B\alpha_0' & C\alpha_0'' \\ A\beta_0 & B\beta_0' & C\beta_0'' \\ L\gamma_0 & M\gamma_0' & N\gamma_0'' \end{vmatrix} = LBC\gamma_0 (\alpha_0'\beta_0'' - \alpha_0''\beta_0') +$$
$$+ MCA\gamma_0' (\alpha_0''\beta_0 - \alpha_0\beta_0'') + NAB\gamma_0'' (\alpha_0\beta_0' - \alpha_0'\beta_0)$$

But

$$\gamma = \alpha'\beta'' - \alpha''\beta', \qquad \gamma' = \alpha''\beta - \alpha\beta'', \qquad \gamma'' = \alpha\beta' - \alpha'\beta$$

Therefore,

$$\Delta = LBC\gamma_0^2 + MCA\gamma_0'^2 + NAB\gamma_0''^2 \neq 0$$

and

$$\begin{aligned} \dot{p}_0 &= \frac{\rho F}{\Delta} (CM\alpha_0''\gamma_0' - BN\alpha_0'\gamma_0'') \\ \dot{q}_0 &= \frac{\rho F}{\Delta} (AN\alpha_0\gamma_0'' - CL\alpha_0''\gamma_0) \\ \dot{r}_0 &= \frac{\rho F}{\Delta} (BL\alpha_0'\gamma_0 - AM\alpha_0\gamma_0') \end{aligned}$$

Substituting these values into (28), we obtain

$$\nu = \frac{\rho F}{\Delta} [\beta_0 (CM\alpha_0''\gamma_0' - BN\alpha_0'\gamma_0'') + \beta_0' (AN\alpha_0\gamma_0'' - CL\alpha_0''\gamma_0) +$$
$$+ \beta_0'' (BL\alpha_0'\gamma_0 - AM\alpha_0\gamma_0')]$$

which on the substitution of (26) finally yields

$$\nu = \rho^2 F \frac{MN\beta_0^2 + NL\beta_0'^2 + LM\beta_0''^2 + m\rho^2 (L\gamma_0^2 + M\gamma_0'^2 + N\gamma_0''^2)}{LBC\gamma_0^2 + MCA\gamma_0'^2 + NAB\gamma_0''^2}$$

Hence the center of the sphere moves with uniform acceleration in the direction of the line of steepest descent, and the acceleration is a function of the initial orientation, with respect to the plane, of the principal axes of the central ellipsoid of inertia of the sphere.

Substituting for ξ_1 from (23) into (20) and (21) yields, on the assumed conditions,

$$\begin{gathered}
\frac{1}{H} (MN\dot\gamma^2 + NL\dot\gamma'^2 + LM\dot\gamma''^2) + m\rho^2 (\dot\gamma^2 + \dot\gamma'^2 + \dot\gamma''^2) = \nu F t^2 \\
\frac{1}{H^2} [(LN\gamma''\dot\gamma' - LM\gamma'\dot\gamma'')^2 + (ML\gamma\dot\gamma'' - MN\gamma''\dot\gamma)^2 + \\
+ (NM\gamma'\dot\gamma - NL\gamma\dot\gamma')^2] + \frac{m\rho^2}{H} (MN\dot\gamma'^2 + NL\dot\gamma'^2 + LM\dot\gamma''^2) = \rho^2 F (F - m\nu) t^2
\end{gathered} \tag{29}$$

We introduce a new independent variable τ by the equation

$$2t\, dt = d\tau$$

Then (29) takes the form

$$\begin{gathered}
\frac{1}{H}\left[MN\left(\frac{d\gamma}{d\tau}\right)^2 + NL\left(\frac{d\gamma'}{d\tau}\right)^2 + LM\left(\frac{d\gamma''}{d\tau}\right)^2\right] + m\rho^2\left[\left(\frac{d\gamma}{d\tau}\right)^2 + \left(\frac{d\gamma'}{d\tau}\right)^2 + \left(\frac{d\gamma''}{d\tau}\right)^2\right] = \frac{1}{4}\nu F \\
\frac{1}{H^2}\left[\left(LN\gamma''\frac{d\gamma'}{d\tau} - LM\gamma'\frac{d\gamma''}{d\tau}\right)^2 + \left(ML\gamma\frac{d\gamma''}{d\tau} - MN\gamma''\frac{d\gamma}{d\tau}\right)^2 + \right. \\
\left. + \left(NM\gamma'\frac{d\gamma}{d\tau} - NL\gamma\frac{d\gamma'}{d\tau}\right)^2\right] + \frac{m\rho^2}{H}\left[MN\left(\frac{d\gamma}{d\tau}\right)^2 + NL\left(\frac{d\gamma'}{d\tau}\right)^2 + LM\left(\frac{d\gamma''}{d\tau}\right)^2\right] = \\
= \frac{\rho^2 F}{4}(F - m\nu)
\end{gathered} \tag{30}$$

Equations (30) coincide with equations (18), (19) of Chaplygin's paper [1], which describe the motion of a sphere on a horizontal plane on the assumption that the angular momentum is horizontal.

Hence the solution of our problem has been reduced to a problem already investigated. All the results obtained by Chaplygin in his work for $h = 0$ can be extended to our problem if the time is replaced by the parameter τ.

In particular, it remains to correct the geometric interpretation of the motion given by Chaplygin in his paper, which consists in the following: there are associated with the sphere two quadric surfaces - an ellipsoid and a hyperboloid. Describe a square about the sphere with one side of the square tangent to the sphere at its point of contact with the plane and perpendicular to the line of steepest descent, and with the side parallel to the first side tangent to the sphere at the diametrically opposite point. The surfaces associated with the sphere touch the sides of this square during the motion of the sphere. As distinct from the problem considered by Chaplygin the square does not move with constant velocity in the direction perpendicular to its plane, but with uniform acceleration ν.

In the general case, i.e. for $n \neq 0$, $h \neq 0$, the problem reduces to the integration of a system of three ordinary differential equations of the first order.

BIBLIOGRAPHY

1. Chaplygin, S.A., O katanii shara po gorizontal'noi ploskosti (On the rolling of a sphere on a horizontal plane). *Sobr. soch.* Vol. 1. Gostekhizdat, Moscow, Leningrad, 1948.

2. Golubev, V.V., *Lektsii po integrirovaniiu uravnenii dvizheniia (Lectures on the Integration of the Equations of Motion)*. p. 27, 1953.

Translated by H.K.